Can torsion play a role in angular momentum conservation law?

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Abstract

In Einstein-Cartan theory, by the use of the general Noether theorem, the general covariant angular-momentum conservation law is obtained with the respect to the local Lorentz transformations. The corresponding conservative Noether current is interpreted as the angular momentum tensor of the gravity-matter system including the spin density. It is pointed out that, assuming the tetrad transformation given by eq. (15), torsion tensor can not play a role in the conservation law of angular momentum.

KEY WORDS: Torsion, Conservation Law, Angular momentum

1 Introduction

Conservation laws of energy-momentum and angular momentum have been of fundamental interest in gravitational physics[1]. Using the vierbein representation of general relativity, Duan (one of the present author) *et al* obtained a general covariant conservation law of energy-momentum which overcomes the difficulties of other expressions[2]. This conservation law gives the correct quadrupole radiation formula of energy which is in good agreement with the analysis of the gravitational damping for the pulsar PSR1916-13[3]. Also,

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from the same point of view, Duan and Feng[4] proposed a covariant conservation law of angular momentum in Riemann space–time which does not suffer from the flaws of the others.[5, 6, 7]

On the other hand, though the Einstein theory of general relativity has succeeded in many respects, there is an essential difficulty in this theory: we could not get a successful renormalized quantum gravity theory[8]. In order to find renormalized theories, many physicists[9, 10] have studied this problem in its more general aspects, i.e. extending Einstein's theory to Einstein-Cartan theory, which includes torsion[11].

As is well known, torsion is a slight modification of the Einstein's theory of relativity[12], but is a generalization that appears to be necessary when one tries to conciliate general relativity with quantum theory. Like opening a Pandora's box, many works have been done in this region[13, 14]. Today, general relativity with non-zero torsion is a major contender for a realistive generalization of the theory of gravitation.

About two decades ago, Hehl[11] gave, in Einstein-Cartan theory, an expression of the angular momentum conservation law which was worked out from Noether's theorem, but in that expression, all quantities carried Riemannian indices and the total angular momentum depended on the coordinative choice which is not an observable quantity. Some physicists[9, 11] investigated the same problem from the local Poincaré transformation and presented another expression of conservation law which is not general covariant, and they did not provide the superpotentials which is much more important in conservation law, hence this theory cannot be said to be a very satisfactory theory. Recently, Hammond[15] obtained his expression of the angular momentum conservation law, unfortunately this theory also met with the difficulties mentioned above.

Several years ago, the general covariant energy-momentum conservation law in general space-time has been discussed successfully by Duan et.al[16]. In this paper, we will study the angular momentum conservation law in Einstein-Cartan theory via the vierbein representation. General relativity without vierbein is like a boat without a jib–without these vital ingredients the going is slow and progress inhibited. Consequently, vierbein have grown to be an indispensable tool in many aspects of general relativity. More important, it is relevant to the physical observability[17]. Based on the Einstein's observable time and space interval, we take the local point of view that any measurement in physics is performed in the local flat reference system whose

existence is guaranteed by the equivalence principle, i.e. an observable object must carries the indices of the internal space. Thus, we draw the support from vierbein not only for mathematical reasons, but also because of physical measurement consideration.

This paper is organized as follows: In section 2, we discuss the general conservation laws in general case. In section 3, by making use of the general Noether theorem, we obtain, in Einstein–Cartan theory, the general covariant angular-momentum conservation law with the respect to the local Lorentz transformations. The corresponding conservative Noether current is interpreted as the angular momentum tensor of the gravity-matter system including the spin density. In section 4, we give a brief summary of the above discussion and point out that torsion tensor can not play a role in the conservation law of angular momentum.

2 Conservation law in general case

The conservation law is one of the important essential problems in gravitational theory. It is due to the invariance of Lagrangian corresponding to some transformations. In order to study the covariant angular momentum conservation law, it is necessary to discuss conservation law by the Noether theorem in general case.

The action of a system is

$$I = \int_M \mathcal{L}(\phi^A, \phi^A_{,\mu})\, d^4x, \tag{1}$$

where ϕ^A are independent variables with general index A, $\phi^A_{,\mu} = \partial_\mu \phi^A$. If the action is invariant under the infinitesimal transformation

$$x'^\mu = x^\mu + \delta x^\mu, \tag{2}$$

$$\phi'^A(x') = \phi^A(x) + \delta\phi^A(x), \tag{3}$$

and $\delta\phi^A$ is zero on the boundary of the four-dimensional volume M, then we can prove that there is the relation

$$\frac{\partial}{\partial x^\mu}(\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi^A_{,\mu}}\delta_0\phi^A) + [\mathcal{L}]_{\phi^A}\delta_0\phi^A = 0, \tag{4}$$

where $[\mathcal{L}]_{\phi^A}$ is

$$[\mathcal{L}]_{\phi^A} = \frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu (\frac{\partial \mathcal{L}}{\partial \phi^A_{,\mu}}),$$

and $\delta_0 \phi^A$ is the Lie derivative of ϕ^A

$$\delta_0 \phi^A = \delta \phi^A(x) - \phi^A_{,\mu} \delta x^\mu. \tag{5}$$

If $\mathcal{L}$ is the total Lagrangian density of the system, there is $[\mathcal{L}]_{\phi^A} = 0$, the field equation of ϕ^A with respect to $\delta I = 0$. From (4) we know that there is a conservation equation corresponding to transformations (2) and (3)

$$\frac{\partial}{\partial x^\mu}(\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi^A_{,\mu}} \delta_0 \phi^A) = 0. \tag{6}$$

This is just the conservation law in general case. It must be pointed out that if $\mathcal{L}$ is not the total Lagrangian density of the system, then as long as the action of $\mathcal{L}$ remains invariant under transformations (2) and (3), (4) is still tenable. But (6) is not admissible now due to $[\mathcal{L}]_{\phi^A} \neq 0$.

In gravitational theory with the vierbein as element fields we can separate ϕ^A as $\phi^A = (e^a_\mu, \psi^B)$, where e^a_μ is the vierbein field and ψ^B is an arbitrary tensor under general coordinate transformation. When ψ^B is $\psi^{\mu_1 \mu_2 \dots \mu_k}$, we can always scalarize it by

$$\psi^{a_1 a_2 \dots a_k} = e^{a_1}_{\mu_1} e^{a_2}_{\mu_2} \dots e^{a_k}_{\mu_k} \psi^{\mu_1 \mu_2 \dots \mu_k},$$

so we can take ψ^B as a scalar field under general coordinate transformations. In later discussion we can simplify the equations by such a choice.

3 Angular momentum conservation law in Einstein–Cartan theory

In Einstein-Cartan theory, the total action of the gravity$-$matter system is expressed as[11]

$$I = \int_M \mathcal{L} d^4x = \int_M (\mathcal{L}_g + \mathcal{L}_m) d^4x, \tag{7}$$

$$\mathcal{L}_g = \frac{c^4}{16\pi G} \sqrt{-g} R.$$

$\mathcal{L}_g$ is the gravitational Lagrangian density, R is the scalar curvature of the Riemann–Cartan space–time. The matter part Lagrangian density $\mathcal{L}_m$ take the form $\mathcal{L}_m = \mathcal{L}_m(\phi^A, D_\mu\phi^A)$, where the matter field ϕ^A belongs to some representation of Lorentz group whose generators are I_{ab} $(a, b = 0, 1, 2, 3)$ and $I_{ab} = -I_{ba}$, D_μ is the covariant derivative operator of ϕ^A

$$D_\mu\phi^A = \partial_\mu\phi^A - \frac{1}{2}\omega_{\mu ab}(I_{ab})^A_B\phi^B.$$

As in E$-$C theory, the affine connection $\Gamma^\lambda_{\mu\nu}$ is not symmetry in μ and ν, i.e. there exists non-zero torsion tensor

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}.$$

It is well known, for vierbein field e^a_μ, the total covariant derivative is equal to zero, i.e.

$$\mathcal{D}_\mu\, e^a_\nu \equiv \partial_\mu\, e^a_\nu - \omega_{\mu ab}\, e^b_\nu - \Gamma^\lambda_{\mu\nu}\, e^b_\lambda = 0. \tag{8}$$

This formula is also can be looked upon as the definition of the spin connection $\omega_{\mu ab}$. From (8), we can get the total expansion of the spin connection with vierbein and torsion

$$\omega_{abc} = \bar{\omega}_{abc} + \frac{1}{2}(T_{cab} + T_{acb} + T_{bca}),$$

$$\bar{\omega}_{abc} = e^\mu_a(\partial_\mu e^\nu_b + \{^{\ \nu}_{\mu\sigma}\}e^\sigma_b)e_{\nu c}, \tag{9}$$

where $\omega_{abc} = e^\mu_a\omega_{\mu bc}$, $T_{abc} = e_{\lambda a}\, e^\mu_b\, e^\nu_c T^\lambda_{\mu\nu}$ are, respectively, the representation of spin connection and torsion tensor in vierbein theory, e^μ_a is the inverse of e^a_μ, $\{^{\ \nu}_{\mu\sigma}\}$ is the Christoffel symbol. By tedious calculation, such decomposition of ω_{abc} allows us to obtain the identity[16]

$$\mathcal{L}_g = \frac{c^4}{16\pi G}\sqrt{-g}R = \frac{c^4}{16\pi G}\Delta - \mathcal{L}_{\bar{\omega}} + \mathcal{L}_T - \frac{c^4}{8\pi G}\mathcal{L}_{\partial T}, \tag{10}$$

where

$$\mathcal{L}_{\bar{\omega}} = \frac{c^4}{16\pi G}\sqrt{-g}(\bar{\omega}_a\bar{\omega}_a - \bar{\omega}_{abc}\bar{\omega}_{cba}), \quad \bar{\omega}_a = \bar{\omega}_{bab}, \tag{11}$$

$$\Delta = \partial_\mu(\sqrt{-g}(e^\mu_a\partial_\nu e^\nu_a - e^\nu_a\partial_\nu e^\mu_a), \tag{12}$$

$$\mathcal{L}_T = \frac{c^4}{16\pi G}\sqrt{-g}(T_aT_a - \frac{1}{2}T_{abc}T_{cba} - \frac{1}{4}T_{abc}T_{abc}), \;\; T_a = T_{bab}, \tag{13}$$

$$\mathcal{L}_{\partial T} = \partial_\mu(\sqrt{-g}e^\mu_a T_a). \tag{14}$$

It is well known that in deriving the general covariant conservation law of energy momentum in general relativity, the general displacement transformation, which is a generalization of the displacement transformation in the Minkowski space-time, was used[18]. In the local Lorentz reference frame, the general displacement transformation takes the same form as that in the Minkowski space-time. This implies that general covariant conservation laws are corresponding to the invariance of the action under local transformations. We may conjecture that since the conservation law for angular momentum in special relativity corresponds to the invariance of the action under the Lorentz transformation, the general covariant conservation law of angular momentum in general relativity may be obtained by means of the local Lorentz invariance.

We choose vierbein e^a_μ, torsion T_{abc} and the matter field ϕ^A as independent variables. Under the local Lorentz transformation

$$e^a_\mu(x) \to e'^a_\mu(x) = \Lambda^a{}_b(x) e^b_\mu(x), \quad \Lambda^a{}_c(x)\Lambda^c{}_b(x) = \delta^a_b, \tag{15}$$

T_{abc} and ϕ^A transform as

$$T_{abc} \to T'_{abc}(x) = \Lambda^l{}_c(x)\Lambda^m_c(x)\Lambda^n{}_c(x) T_{lmn}, \tag{16}$$

$$\phi^A \to \phi'^A(x) = D(\Lambda(x))^A{}_B \phi^B(x). \tag{17}$$

Since the coordinates x^μ do not transform under the local Lorentz transformation, $\delta x^\mu = 0$, from (5), it can be proved that in this case, $\delta_0 \to \delta$. It is required that $\mathcal{L}_m$ is invariant under (15) and $\mathcal{L}_g$ is invariant obviously. So under the local Lorentz transformation (15) $\mathcal{L}$ is invariant. In the light of the discussion in section 2, we would like to have the relation

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu e^\nu_a}\delta e^\nu_a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^A}\delta\phi^A + \frac{\partial \mathcal{L}}{\partial \partial_\mu T_{abc}}\delta T_{abc}\right) + [\mathcal{L}]_{e^\nu_a}\delta e^\nu_a + [\mathcal{L}]_{T_{abc}}\delta T_{abc} + [\mathcal{L}]_{\phi^A}\delta\phi^A = 0, \tag{18}$$

where $[\mathcal{L}]_{e^\nu_a}$, $[\mathcal{L}]_{T_{abc}}$ and $[\mathcal{L}]_{\phi^A}$ are the Euler expressions defined as

$$[\mathcal{L}]_{e^\nu_a} = \frac{\partial \mathcal{L}}{\partial e^\nu_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu e^\nu_a},$$

$$[\mathcal{L}]_{T_{abc}} = \frac{\partial \mathcal{L}}{\partial T_{abc}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu T_{abc}},$$

$$[\mathcal{L}]_{\phi^A} = \frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^A}.$$

Using the Einstein equation $[\mathcal{L}]_{e^\nu_a} = 0$, Einstein-Cartan equation $[\mathcal{L}]_{T_{abc}} = 0$ and the equation of motion of matter $[\mathcal{L}]_{\phi^A} = 0$, we get following by (18)

$$\frac{\partial}{\partial x^\mu}(\frac{\partial \mathcal{L}_g}{\partial \partial_\mu e^\nu_a}\delta e^\nu_a + \frac{\partial \mathcal{L}_g}{\partial \partial_\mu T_{abc}}\delta T_{abc}) + \frac{\partial}{\partial x^\mu}(\frac{\partial \mathcal{L}_m}{\partial \partial_\mu e^\nu_a}\delta e^\nu_a + \frac{\partial \mathcal{L}_m}{\partial \partial_\mu \phi^A}\delta \phi^A) = 0, \quad (19)$$

where we have used the fact that only $\mathcal{L}_m$ contains the matter field ϕ^A, and with its structure $\mathcal{L}_m$ does not possess $\partial_\mu T_{abc}$. Consider the infinitesimal local Lorentz transformation $\Lambda^a_{\ b}(x) = \delta^a_b + \alpha^a_{\ b}(x)$, $\alpha_{ab} = -\alpha_{ba}$, $D(\Lambda)$ can be linearized as $[D(\Lambda)]^A_{\ B} = \delta^A_B + \frac{1}{2}(I_{ab})^A_{\ B}\alpha_{ab}$, we have

$$\begin{aligned} \delta e^\nu_a(x) &= \alpha_{ab} e^\nu_b(x), \\ \delta T_{abc}(x) &= \alpha_{ad} T_{dbc}(x) + \alpha_{bd} T_{adc}(x) + \alpha_{cd} T_{abd}(x), \quad (20) \\ \delta \phi^A(x) &= \frac{1}{2}(I_{ab})^A_{\ B}\phi^B(x)\alpha_{ab}(x). \end{aligned}$$

We introduce j^μ_{ab}

$$\sqrt{-g} j^\mu_{ab}\alpha_{ab} = \frac{3}{c}[\frac{\partial \mathcal{L}_{\bar{\omega}}}{\partial \partial_\mu e^\nu_a} e^\nu_b \alpha_{ab} - \frac{\partial \mathcal{L}_m}{\partial \partial_\mu e^\nu_a} e^\nu_b \alpha_{ab} - \frac{1}{2}\frac{\partial \mathcal{L}_m}{\partial \partial_\mu \phi^A}(I_{ab})^A_{\ B}\phi^B \alpha_{ab}], \quad (21)$$

then (19) can be rewritten as

$$\partial_\mu(\sqrt{-g} j^\mu_{ab}\alpha_{ab}) - \frac{3c^3}{16\pi G}\partial_\mu(\frac{\partial \Delta}{\partial \partial_\mu e^\nu_a} e^\nu_b \alpha_{ab}) + \frac{3c^3}{8\pi G}\partial_\mu(\frac{\partial \mathcal{L}_{\partial T}}{\partial \partial_\mu e^\nu_a} e^\nu_b \alpha_{ab} + \frac{\partial \mathcal{L}_{\partial T}}{\partial \partial_\mu T_{abc}}\delta T_{abc}) = 0. \quad (22)$$

From (12) one can get easily that

$$\frac{\partial \Delta}{\partial \partial_\mu e^\nu_a} e^\nu_b \alpha_{ab} = \alpha_{ab}\partial_\mu(\sqrt{-g} V^{\mu\lambda}_{ab}), \quad (23)$$

where

$$V^{\mu\lambda}_{ab} = e^\mu_a e^\lambda_b - e^\mu_b e^\lambda_a. \quad (24)$$

Now, let us investigate the third term in left-hand side (LHS) of (22). With (14), we can get

$$\frac{\partial \mathcal{L}_{\partial T}}{\partial \partial_\mu e^\nu_a} e^\nu_b \alpha_{ab} = \sqrt{-g} T_a e^\mu_b \alpha_{ab}, \quad (25)$$

and from (14) and (20), we also have

$$\frac{\partial \mathcal{L}_{\partial T}}{\partial \partial_\mu T_{abc}} \delta T_{abc} = \sqrt{-g} e_a^\mu T_b \alpha_{ab}. \tag{26}$$

Considering that α_{ab} is antisymmetry, i.e. $\alpha_{ab} = -\alpha_{ba}$, we draw the conclusion from (25) and (26) that the third term in LHS of (22) is equal to zero. Substituting (23), (25) and (26) into (22), we obtain

$$\partial_\mu(\sqrt{-g} j_{ab}^\mu)\alpha_{ab} + [\sqrt{-g} j_{ab}^\mu - (\frac{3c^3}{16\pi G})\partial_\nu(\sqrt{-g} V_{ab}^{\nu\mu})]\partial_\mu \alpha_{ab} = 0. \tag{27}$$

Since α_{ab} and $\partial_\mu \alpha_{ab}$ are independent of each other, we must have

$$\partial_\mu(\sqrt{-g} j_{ab}^\mu) = 0, \tag{28}$$

$$j_{ab}^\mu = \frac{3c^3}{16\pi G} \frac{1}{\sqrt{-g}} \partial_\nu(\sqrt{-g} V_{ab}^{\nu\mu}), \tag{29}$$

or

$$j_{ab}^\mu = \frac{3c^3}{16\pi G}(\bar{\omega}_a e_b^\mu + \bar{\omega}_{abc} e_c^\mu - \bar{\omega}_b e_a^\mu - \bar{\omega}_{bac} e_c^\mu). \tag{30}$$

From (28) and (29), it can be concluded that j_{ab}^μ is conserved identically. As usual, we call $V_{ab}^{\nu\mu}$ superpotentials. Since the current j_{ab}^μ is derived from the local Lorentz invariance of the total Lagrangian, it can be interpreted as the total angular momentum tensor density of the gravity-matter system, and it contains the spin density of the matter field: $(\partial \mathcal{L}_m)/(\partial \partial_\mu e_a^\nu)$. From the above discussion, we see that not only the current j_{ab}^μ but also the superpotential $V_{ab}^{\mu\nu}$ do not have any terms relevant to torsion tensor, all of them are only determined by the vierbein.

For a globally hyperbolic Riemann-Cartan manifold, there exist Cauchy surfaces Σ_t foliating M. We choose a submanifold D of M joining any two Cauchy surfaces Σ_{t_1} and Σ_{t_2} so the boundary ∂D of D consists of three parts $\Sigma_{t_1}, \Sigma_{t_2}$ and A which is at spatial infinity. For an isolated system, the space-time should be asymptotically flat at spatial infinity, so the vierbein have the following asymtotical behavior[19]

$$\lim_{r\to\infty}(\partial_\mu e_{\nu a} - \partial_\nu e_{\mu a}) = 0 \tag{31}$$

Since

$$\sqrt{-g} V_{ab}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\epsilon_{abcd} e_{\lambda c} e_{\rho d},$$

we have $\lim_{r\to\infty}\partial_\lambda(\sqrt{-g}V_{ab}^{\lambda\mu})=0$. Thus, we can get the total conservative angular momentum from (28) and (29)

$$J_{ab}=\int_{\Sigma_t} j_{ab}^{\mu}\sqrt{-g}d\Sigma_\mu=\frac{3c^3}{16\pi G}\int_{\partial\Sigma_t}\sqrt{-g}V_{ab}^{\mu\nu}d\sigma_{\mu\nu}, \tag{32}$$

where $\sqrt{-g}d\Sigma_\mu$ is the covariant surface element of Σ_t, $d\Sigma_\mu=\frac{1}{3!}\epsilon_{\mu\nu\lambda\rho}dx^\nu\wedge dx^\lambda\wedge dx^\rho$, $d\sigma_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}dx^\lambda\wedge dx^\rho$.

4 Discussion

In summary, we have succeeded in obtaining an expression of an angular momentum conservation law in Riemann–Cartan space–time. This conservation law has the following main properties:

1. It is a covariant theory with respect to the generalized coordinate transformations, but the angular momentum tensor is not covariant under the local Lorentz transformation which, due to the equivalent principle, is reasonable to require.

2. For a closed system, the total angular momentum does not depend on the choice of the Riemannian coordinates and, according (31), the space-time at spatial infinity is flat, thus the conservative angular momentum J_{ab} should be a covariant object when we make a Lorentz transformation $\Lambda_{ab}=A_{ab}=$ constant at spatial infinity, as in special relativity

$$J'_{ab}=A^c{}_a A^d{}_b J_{cd}.$$

To understand this the key point is that to obtain J_{ab}, one has to enclose everything of the closed system, and every point of space-time at spatial infinity belongs to the same Minkowski space-time in that region. This means that in general relativity for a closed system, the total angular momentum J_{ab} must be looked upon as a Lorentz tensor like that in spatial relativity.

3. The conservative angular-momentum current and the corresponding superpotential in Einstein–Cartan theory are the same with those in Einstein theory[4], torsion can not play a role in the conservation law. Both angular-momentum current and the superpotential are determined only by vierbein field. This result makes us be flabbergasted. As is well known, the torsion tensor is closely related to the spin density tensor through the famous

Einstein–Cartan field equation, it is natural for us to believe that the torsion tensor should play some important role in the conservation law. Is that true? As mentioned in this paper, with the accurate calculation, the answer is no. Then a question arises: Why can not the torsion play a role in the angular momentum conservation law? We hope we can work it out in the future.

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